

Covariant Stückelberg analysis of de Rham-Gabadadze-Tolley massive gravity with a general fiducial metric

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The Stückelberg analysis of nonlinear massive gravity in the presence of a general fiducial metric is investigated. We develop a “covariant” formalism for the Stückelberg expansion by working with a local inertial frame, through which helicity modes can be characterized correctly. Within this covariant approach, an extended Λ_3 decoupling limit analysis can be consistently performed, which keeps $\bar{R}_{\mu\nu\rho\sigma}/m^2$ fixed with $\bar{R}_{\mu\nu\rho\sigma}$ the Riemann tensor of the fiducial metric. In this extended decoupling limit, the scalar mode π acquires self-interactions due to the presence of the curvature of the fiducial metric. However, the equation of motion for π remains of second order in derivatives, which extends the understanding of the absence of the Boulware Deser ghost in the case of a flat fiducial metric.

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I. INTRODUCTION

Massive gravity is a candidate of modified gravity which explains the current accelerated expansion of the Universe [1, 2]. A linear theory of massive graviton was first considered by Fierz and Pauli (FP) [3]. This theory succeeded in excluding the extra ghost degree of freedom at linear order, but van Dam, Veltman, and Zakharov suggested that this theory does not reduce to general relativity (GR) even in the massless limit [4, 5]. However, Vainshtein showed that this problem is caused by omitting the nonlinear effects [6]. For this reason, nonlinear extensions of the FP theory have been actively considered. However, for a long time, these nonlinear theories had suffered from the Boulware Deser (BD) ghost problem [7], which states that the theories have the extra ghost degree of freedom in addition to the usual 5 degrees of freedom of massive spin-2.

In order to see the BD ghost explicitly, the Stückelberg formalism is very useful [8]. In Stückelberg language, the physical degrees of freedom are decomposed into helicity-0, helicity-1, and helicity-2 modes. The origin of the BD ghost mode is understood as the higher order equation of motion for the helicity-0 mode. In Refs. [11, 12], de Rham, Gabadadze, and Tolley (dRGT) constructed the mass potential in which the self-interactions of helicity-0 mode is tuned to be a total divergence, leaving a second-order equation of motion for the helicity-0 mode. It can be seen that after a field redefinition the action for the helicity-0 mode indeed reduces to the Galileon form [13] clearly having the second-order equation of motion. Thus, this theory is BD ghost free. Massive gravity with this mass potential is called the dRGT theory. Hassan and Rosen investigated dRGT massive gravity by means of the Hamiltonian analysis, and proved that the dRGT theory is free of the BD ghost even away from the decoupling limit [14, 16]. In addition, they generalized dRGT massive gravity on a flat fiducial metric to the theory on a general fiducial metric, and proved that the theory is also free of the BD ghost using the same method [15, 16].

Although formal Hamiltonian analysis is sufficient to prove the absence of ghost, it is interesting to show this explicitly at the level of equations of motion in a Stückelberg language. In a complementary way to the Hamiltonian analysis of Hassan and Rosen, the Stückelberg analysis in the flat fiducial case has been studied in detail in Refs. [17–20]. Moreover, the Stückelberg analysis was extended to the curved fiducial case: de Rham and Renaux-Petel studied the de Sitter fiducial case [21], while Fasiello and Tolley analyzed the Friedmann-Lemaître-Robertson-Walker fiducial case [22]. However, the Stückelberg analysis has not been performed so far in the general fiducial case. (Hamiltonian analysis in Stückelberg language is discussed in Ref. [23].) In this paper, we discuss the Stückelberg analysis in the dRGT theory with a completely general fiducial metric. First, we extend the definition of the perturbation of the Stückelberg field in a covariant manner. Using this definition, we expand the action in terms of the perturbed quantities up to fourth order. Next, we extend the decoupling limit of the flat case to that of a curved one by scaling

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the curvature scale. Finally, we show that, in this extended decoupling limit, the equation of motion for the helicity-0 mode π does not include higher derivatives of π , which clarifies in a different and complementary way the reason why the BD ghost is absent even in the curved fiducial case.

This paper is organized as follows. In the next section, we briefly review dRGT massive gravity and the usual Stückelberg analysis. In Sec. III, we develop a covariant Stückelberg analysis with a general fiducial metric and apply it to dRGT massive gravity. In Sec. IV, we derive the decoupling limit at the energy scale Λ_3 and show that the equation of motion for π remains of second order in derivatives. The final section is devoted to the conclusions.

II. MASSIVE GRAVITY WITH A CURVED FIDUCIAL METRIC

The Lagrangian for nonlinear massive gravity is composed of the Einstein-Hilbert term and the dRGT mass terms:

$$S = \frac{M_{\text{pl}}^2}{2} \int d^4x \sqrt{-g} (R + m^2 \mathcal{L}^{\text{dRGT}}), \quad (1)$$

where m is the mass of the graviton $h_{\mu\nu}$ defined by

$$h_{\mu\nu} \equiv g_{\mu\nu} - g_{\mu\nu}^{(0)}, \quad (2)$$

with a fixed background metric $g_{\mu\nu}^{(0)}$. The dRGT mass terms are given by

$$\mathcal{L}^{\text{dRGT}} = \mathcal{U}_2 + \alpha_3 \mathcal{U}_3 + \alpha_4 \mathcal{U}_4, \quad (3)$$

where

$$\mathcal{U}_2 = [\mathcal{K}]^2 - [\mathcal{K}^2], \quad (4)$$

$$\mathcal{U}_3 = [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3], \quad (5)$$

$$\mathcal{U}_4 = [\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2] + 8[\mathcal{K}][\mathcal{K}^3] + 3[\mathcal{K}^2]^2 - 6[\mathcal{K}^4], \quad (6)$$

with α_3 and α_4 being free constant parameters. The square brackets “ $[\mathcal{M}]$ ” denote the trace of the matrix \mathcal{M} with respect to the physical metric $g_{\mu\nu}$. The matrix $\mathcal{K}^\mu{}_\nu$ is defined by

$$\mathcal{K}^\mu{}_\nu = \delta^\mu{}_\nu - \sqrt{g^{\mu\rho} \bar{g}_{\rho\nu}}, \quad (7)$$

where $\bar{g}_{\mu\nu}$ is a fixed symmetric matrix that is called a fiducial metric. Generally speaking, a background metric $g_{\mu\nu}^{(0)}$ on which we define a graviton has nothing to do with the fiducial metric $\bar{g}_{\mu\nu}$, though in many cases $\bar{g}_{\mu\nu}$ is indeed a solution of the background equation of motion. In this paper, we take $g_{\mu\nu}^{(0)} = \bar{g}_{\mu\nu}$ for simplicity, which enables us to expand the matrix \mathcal{K} perturbatively.

A. Stückelberg trick

The matrix $\mathcal{K}^\mu{}_\nu$ and thus the dRGT mass terms (4)–(6) explicitly break general covariance due to the presence of a fixed fiducial metric $\bar{g}_{\mu\nu}$. On the other hand, $\bar{g}_{\mu\nu}$ can always be thought of as the “gauge-fixed” version of some covariant tensor field, which can be constructed using the well-known Stückelberg trick [8–10]:

$$\bar{g}_{\mu\nu}(x) \rightarrow f_{\mu\nu}(x) = \bar{g}_{ab}(\phi(x)) \frac{\partial \phi^a(x)}{\partial x^\mu} \frac{\partial \phi^b(x)}{\partial x^\nu}, \quad (8)$$

where a set of four (we are working in four-dimensional spacetime) Stückelberg fields $\{\phi^a\}$ transform as scalars under a general coordinate transformation of spacetime. The fixed $\bar{g}_{\mu\nu}$ can be recovered by choosing the so-called “unitary gauge” with $\phi^\mu = x^\mu$. By replacing $\bar{g}_{\mu\nu}$ with the covariant tensor field $f_{\mu\nu}$, the dRGT mass terms are promoted to scalars and thus the corresponding Lagrangian (1) acquires general covariance.

Degrees of freedom in a gravity theory alternative to GR show themselves in a simpler manner in the so-called “decoupling limit,” where different types (e.g., helicities) of degrees of freedom decouple from each other in some limit of energy scales. In the case of massive gravity, the decoupling limit is taken as $M_{\text{pl}} \rightarrow \infty$ so that the nonlinearities in gravity get reduced, while keeping the energy scale $\Lambda_\lambda \equiv (M_{\text{pl}} m^{\lambda-1})^\lambda$ with some λ fixed so that interactions arising

above Λ_λ become irrelevant. When a fiducial metric is flat, all degrees of freedom are thus living in the flat Minkowski background, which enables us to identify the (Stückelberg) field-space Lorentz symmetry with the spacetime global Lorentz symmetry, while the later only arises in the decoupling limit. In this case, the set of four Stückelberg fields $\{\phi^a\}$ transform as a vector under this identified global Lorentz transformation. As a result, the fields A_a and π , defined by

$$\phi^a = x^a - \pi^a = x^a - A^a - \partial^a \pi, \quad (9)$$

transform as a vector and a scalar and encode the information of helicity-1 and helicity-0 modes of a massive graviton, respectively, in the decoupling limit. Thus, in this limit, the requirement that the equation of motion for π is of second order in derivatives ensures the absence of BD ghost [10, 28].

The above argument, however, cannot be applied simply to the case of a curved fiducial metric. Indeed, a naive split (9) in Eq. (8) yields

$$f_{\mu\nu} = \bar{g}_{\mu\nu} - \bar{g}_{\rho\mu} \partial_\nu \pi^\rho - \bar{g}_{\rho\nu} \partial_\mu \pi^\rho + \bar{g}_{\rho\sigma} \partial_\mu \pi^\rho \partial_\nu \pi^\sigma, \quad (10)$$

where π^μ is defined as $\pi^a \equiv \delta_\mu^a \pi^\mu$. If we would rewrite all partial derivatives in terms of “covariant” derivatives $\bar{\nabla}$ with respect to $\bar{g}_{\mu\nu}$ using $\partial_\mu \pi^\rho = \bar{\nabla}_\mu \pi^\rho - \bar{\Gamma}_{\mu\lambda}^\rho \pi^\lambda$ and focus on the “supposed-to-be” helicity-0 mode π defined by $\pi^\mu \equiv \bar{g}^{\mu\nu} \partial_\nu \pi$, we will get

$$\begin{aligned} f_{\mu\nu} = & \bar{g}_{\mu\nu} - 2\bar{\nabla}_\mu \bar{\nabla}_\nu \pi + \bar{\nabla}_\mu \bar{\nabla}_\sigma \pi \bar{\nabla}_\nu \bar{\nabla}^\sigma \pi \\ & + \left(\bar{g}_{\rho\mu} \bar{\Gamma}_{\nu\lambda}^\rho + \bar{g}_{\rho\nu} \bar{\Gamma}_{\mu\lambda}^\rho \right) \bar{\nabla}^\lambda \pi \\ & - \bar{\nabla}_\mu \bar{\nabla}_\sigma \pi \bar{\Gamma}_{\nu\tau}^\sigma \bar{\nabla}^\tau \pi - \bar{\nabla}_\nu \bar{\nabla}_\rho \pi \bar{\Gamma}_{\mu\lambda}^\rho \bar{\nabla}^\lambda \pi + \bar{g}_{\rho\sigma} \bar{\Gamma}_{\mu\lambda}^\rho \bar{\Gamma}_{\nu\tau}^\sigma \bar{\nabla}^\lambda \pi \bar{\nabla}^\tau \pi, \end{aligned} \quad (11)$$

which is not explicitly covariant due to the presence of the Christoffel symbols.

In fact, as was well explained in [25] (see also [17, 21]), A_a and π defined in the “naive” split (9) are neither vector nor scalar any longer, and do not capture the helicity-1 and helicity-0 modes correctly, either when going beyond the decoupling limit or when the fiducial metric is curved, which is the case we are dealing with in this paper.

The main purpose of this paper is thus to develop a covariant formalism for the Stückelberg expansion, through which the helicity modes can be characterized correctly and a decoupling limit analysis similar to the case of a flat fiducial metric can be consistently performed.

III. A COVARIANT APPROACH TO STÜCKELBERG ANALYSIS

A. Covariant definition of the perturbed Stückelberg field

For a general fiducial metric, the main difficulty in “covariantly” defining the Goldstone modes π^μ and identifying the helicity modes is that, due to the loss of global symmetries, the identification of the internal and physical space does not make sense any longer. One exception is the case of the maximally symmetric fiducial metric considered in [21], where the identification of the helicity modes was made by embedding the d -dimensional (A)dS into a $(d+1)$ -dimensional Minkowski background and then projecting back. This trick, however, cannot be used for a general fiducial metric since embedding an arbitrary d -dimensional space into a $(d+1)$ -dimensional Minkowski one is not always possible.

In this paper, we employ an alternative approach based on the Riemann normal coordinates (RNC), which is in fact a standard approach to defining perturbations covariantly, as has been used in the well-known background field method (e.g. [24]). See also footnote 5 of [25], in which the use of Riemann normal coordinate is suggested. The idea is to regard the Stückelberg field as the diffeomorphism of the spacetime itself.¹ Precisely, we consider a one-parameter family of diffeomorphisms generated by a set of single-parameter curves $x^\mu(\lambda)$ parametrized by λ , i.e.,

$$\phi_\lambda : p \mapsto \phi_\lambda(p), \quad (12)$$

for a given point p in spacetime. At this point, we do not assume $x^\mu(\lambda)$ to be a geodesic, while we shall see below how the standard RNC approach arises in order to recover the expressions in the case of a flat fiducial metric (9). We

¹ See also Appendix A for an alternative point of view of the field space, which yields exactly the same definition for the Goldstone modes π^μ as in Eq. (21).

may freely set $\lambda = 0$ at a given point p , and define the Stückelberg fields at p as the coordinate values of its image $\phi_\lambda(p)$ at $\lambda = -1$, i.e.,

$$\phi^\mu|_p \equiv x^\mu|_{\phi_{-1}(p)} = x^\mu(-1). \quad (13)$$

Note that in Eq. (13) we use the same symbol both for the diffeomorphism and for the Stückelberg fields. The perturbation of the Stückelberg fields, i.e., the difference between the Stückelberg fields at point p and the coordinate values of point p itself,

$$\phi^\mu|_p - x^\mu|_p = x^\mu(-1) - x^\mu(0), \quad (14)$$

is obviously not a covariant object, since $x^\mu(0)$ are fixed.

We define u^μ as the tangent vector of $x^\mu(\lambda)$,

$$u^\mu(\lambda) \equiv \frac{dx^\mu(\lambda)}{d\lambda}, \quad (15)$$

which is automatically a covariant object by definition. Integrating Eq. (15) gives

$$\begin{aligned} x^\mu(\lambda) &\equiv \left[e^{\lambda \frac{d}{d\lambda'}} x^\mu(\lambda') \right] \Big|_{\lambda'=0} \\ &= \left[\exp \left(\lambda u^\alpha(\lambda') \frac{\partial}{\partial x^\alpha(\lambda')} \right) x^\mu(\lambda') \right] \Big|_{\lambda'=0} \\ &= x_0^\mu + \lambda \xi^\mu + \frac{\lambda^2}{2} \xi^\alpha \frac{\partial u^\mu(\lambda')}{\partial x^\alpha} \Big|_{\lambda'=0} + \frac{\lambda^3}{3!} \xi^\alpha \frac{\partial}{\partial x^\alpha} \left(u^\beta(\lambda') \frac{\partial u^\mu(\lambda')}{\partial x^\beta} \right) \Big|_{\lambda'=0} + \dots, \end{aligned} \quad (16)$$

where we used $\frac{d}{d\lambda} \equiv u^\alpha(\lambda) \frac{\partial}{\partial x^\alpha(\lambda)}$ and denoted

$$\xi^\mu \equiv u^\mu(0), \quad x_0^\mu \equiv x^\mu(0), \quad (17)$$

for short. Setting $\lambda = -1$ in Eq. (16) yields

$$\phi^\mu|_p \equiv x^\mu(-1) = x_0^\mu - \xi^\mu + \frac{1}{2} \xi^\alpha \partial_\alpha \xi^\mu - \frac{1}{3!} \xi^\alpha \partial_\alpha (\xi^\beta \partial_\beta \xi^\mu) + \mathcal{O}(\xi^4). \quad (18)$$

Note that Eq. (16) and thus Eq. (18) are the results of the standard Taylor expansion and we do not assume that $x^\mu(\lambda)$ is a geodesic. If we naively identify ξ^μ as the Goldstone modes, due to the presence of derivatives of ξ^μ , (18) does not reduce to Eq. (9) even in the case of the flat fiducial metric. This discrepancy can be trivially solved by introducing a new variable π^μ through

$$\pi^\mu = \xi^\mu - \frac{1}{2} \xi^\alpha \bar{\nabla}_\alpha \xi^\mu + \frac{1}{3!} \xi^\alpha \bar{\nabla}_\alpha (\xi^\beta \bar{\nabla}_\beta \xi^\mu) + \mathcal{O}(\xi^4), \quad (19)$$

or its inverted form

$$\xi^\mu = \pi^\mu + \frac{1}{2} a^\mu + \frac{1}{12} \pi^\rho \bar{\nabla}_\rho a^\mu + \frac{1}{4} a^\nu \bar{\nabla}_\nu \pi^\mu + \mathcal{O}(\pi^4), \quad (20)$$

where $\bar{\nabla}_\mu$ is the covariant derivative with respect to $\bar{g}_{\mu\nu}$ and $a^\mu \equiv \pi^\nu \bar{\nabla}_\nu \pi^\mu$. Since ξ^μ is covariant, π^μ defined by Eq. (19) is automatically covariant, which we identify as the covariant Goldstone modes. Plugging Eq. (20) into Eq. (18) and performing some manipulation, we have

$$\phi^\mu|_p = x_0^\mu - \pi^\mu - \frac{1}{2} \bar{\Gamma}_{\nu\rho}^\mu \pi^\nu \pi^\rho + \frac{1}{6} (\partial_\nu \bar{\Gamma}_{\rho\sigma}^\mu - 2 \bar{\Gamma}_{\nu\lambda}^\mu \bar{\Gamma}_{\rho\sigma}^\lambda) \pi^\nu \pi^\rho \pi^\sigma + \dots, \quad (21)$$

which indeed reduces to Eq. (9) in the case of the flat fiducial metric. Equation (21) is one of the main results of this paper. This result implies that the covariant Goldstone modes π^μ are nothing but the standard Riemann normal coordinates, which correspond to the tangent vector of the geodesic at point p connecting the point p and its image $\phi_{-1}(p)$. The definitions of the Stückelberg field ϕ^μ and the Goldstone modes π^μ are illustrated in Fig. 1.

Before ending this subsection, we emphasize again that both ξ^μ and π^μ are covariant and have explicit covariant relations (19) or (20), and thus both Eqs. (18) and (21) can be used to derive covariant expressions. We employ Eq. (21) since this has an explicit correspondence to the case of the flat fiducial metric (9).

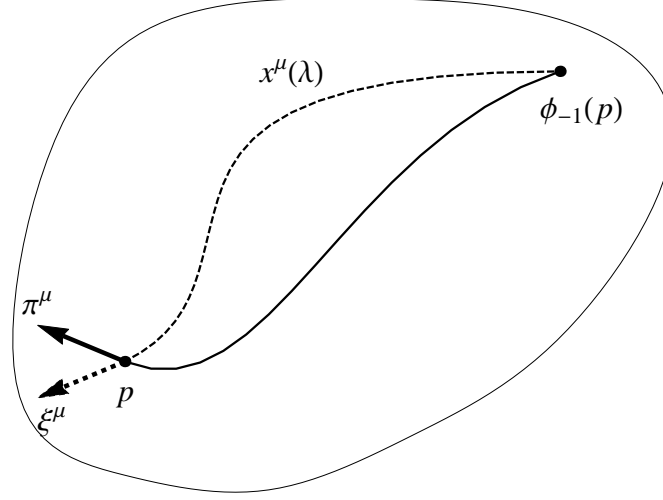


FIG. 1: Illustration of the definitions of Stückelberg field ϕ^μ and the Goldstone modes π^μ . The Stückelberg fields $\phi^\mu|_p$ at a given point p are defined as the coordinate values $x^\mu|_{\phi_{-1}(p)}$, where $\phi_{-1}(p)$ is the image of p under the diffeomorphism generated by single-parameter curves $x^\mu(\lambda)$ with parameter $\lambda = -1$. If $x^\mu(\lambda)$ are not geodesics (dashed curve), the Stückelberg field is expanded as (18); if $x^\mu(\lambda)$ are geodesics (black curve) the Stückelberg field is expanded as (21) with Goldstone modes π^μ as the standard Riemann normal coordinates.

B. Stückelberg expansion of the action

Having defined the perturbative expansion of the Stückelberg and the Goldstone modes π^μ as in Eq. (21), we are able to expand the “covariantized” fiducial metric $f_{\mu\nu}$ in Eq. (8). Simply by plugging Eq. (21) and carefully dealing with the Christoffel symbols and their derivatives, it is straightforward to recast the expressions in terms of π^μ , $\bar{R}_{\mu\nu\rho\sigma}$, and their covariant derivatives (with respect to the fiducial metric). This procedure, however, becomes more and more cumbersome when going to higher orders in π^μ .

In this subsection, we take an equivalent but simpler treatment by recalling that, according to Eq. (8), $f_{\mu\nu}$ can be viewed as the “pull back” of $\bar{g}_{\mu\nu}$ under the diffeomorphism ϕ of spacetime:

$$f_{\mu\nu}|_p = (\phi_{-1}^* \bar{g}_{\mu\nu})|_p, \quad (22)$$

for a given point p . As in the standard lore, an infinitesimal diffeomorphism is generated by a vector field, which is just the tangent vector of the curve $u^\mu = dx^\mu/d\lambda$ in our case. The change in any tensor field induced by such an infinitesimal diffeomorphism is encoded in the Lie derivatives along the curve $x^\mu(\lambda)$. Precisely, we introduce a coordinate system adapted to u^μ , such that the parameter λ along the curve $x^\mu(\lambda)$ is chosen as one of the coordinates, e.g., x^0 , while the values of other coordinates $\{x^i\}$ are kept invariant along the curve. In this particular coordinate system, for a given point p with coordinate values $x^\mu \equiv (x^0, x^i)$, we have $\phi^\mu|_p = (x^0 - 1, x^i)$ and thus

$$\left. \frac{\partial \phi^\mu}{\partial x^\nu} \right|_p = \delta^\mu_\nu. \quad (23)$$

The components of $f_{\mu\nu}$ in this peculiar coordinate system are thus given by

$$\begin{aligned} f_{\mu\nu}(x) \equiv f_{\mu\nu}|_p &= (\phi_{-1}^* \bar{g}_{\mu\nu})|_p \\ &= \bar{g}_{\mu\nu}(x^0 - 1, x^i) \\ &\equiv e^{-\partial/\partial x^0} \bar{g}_{\mu\nu}(x). \end{aligned} \quad (24)$$

On the other hand, in this peculiar coordinate system, $\partial/\partial x^0$ is equivalent to the Lie derivative \mathcal{L}_u when acting on

any tensor, Eq. (24) can be recast into a covariant form²:

$$f_{\mu\nu}(x) = e^{-\mathcal{L}_\xi} \bar{g}_{\mu\nu}(x), \quad (25)$$

where $\xi^\mu \equiv u^\mu|_p$. Equation (25) is also one of the main results of this paper, which now actually holds in any coordinate system.

Using Eq. (25), it is now straightforward to expand $f_{\mu\nu}$ in term of ξ^μ :

$$\begin{aligned} f_{\mu\nu} &= \bar{g}_{\mu\nu} - 2\bar{\nabla}_{(\mu}\xi_{\nu)} \\ &\quad + \bar{\nabla}_\mu \xi_\rho \bar{\nabla}_\nu \xi^\rho - \bar{R}_{\mu\rho\nu\sigma} \xi^\rho \xi^\sigma + \bar{\nabla}_{(\mu} \tilde{a}_{\nu)} \\ &\quad + \frac{1}{3} \bar{\nabla}_\lambda \bar{R}_{\mu\rho\nu\sigma} \xi^\lambda \xi^\rho \xi^\sigma + \frac{2}{3} \bar{R}_{\mu\rho\lambda\sigma} \bar{\nabla}_\nu \xi^\lambda \xi^\rho \xi^\sigma + \frac{2}{3} \bar{R}_{\nu\rho\lambda\sigma} \bar{\nabla}_\mu \xi^\lambda \xi^\rho \xi^\sigma \\ &\quad + \bar{R}_{(\mu|\rho|\nu)\sigma} \tilde{a}^\rho \xi^\sigma - \bar{\nabla}_{(\mu} \xi^\rho \bar{\nabla}_{\nu)} \tilde{a}_\rho - \frac{1}{3} \bar{\nabla}_{(\mu} (\xi^\rho \nabla_\rho \tilde{a}_{|\nu)}) + \mathcal{O}(\xi^4), \end{aligned} \quad (26)$$

where all indices are raised and lowered by $\bar{g}^{\mu\nu}$ and $\bar{g}_{\mu\nu}$, respectively, $\bar{R}^\mu_{\nu\rho\sigma}$ is the Riemann tensor constructed from $\bar{g}_{\mu\nu}$, and $\tilde{a}^\mu \equiv \xi^\nu \bar{\nabla}_\nu \xi^\mu$. It is more convenient to eliminate \tilde{a}^μ by using π^μ defined by Eq. (19) or Eq. (20), which yields

$$\begin{aligned} \tilde{a}^\mu &\equiv \xi^\nu \bar{\nabla}_\nu \xi^\mu \\ &= a^\mu + \frac{1}{2} \pi^\nu \bar{\nabla}_\nu a^\mu + \frac{1}{2} a^\nu \bar{\nabla}_\nu \pi^\mu + \mathcal{O}(\pi^4), \end{aligned} \quad (27)$$

with $a^\mu \equiv \pi^\nu \bar{\nabla}_\nu \pi^\mu$. Plugging Eq. (27) into Eq. (26) and performing simple manipulations, we get

$$f_{\mu\nu} = f_{\mu\nu}^{(0)} + f_{\mu\nu}^{(1)} + f_{\mu\nu}^{(2)} + f_{\mu\nu}^{(3)} + \mathcal{O}(\pi^4), \quad (28)$$

with

$$f_{\mu\nu}^{(0)} = \bar{g}_{\mu\nu}, \quad (29)$$

$$f_{\mu\nu}^{(1)} = -2\bar{\nabla}_{(\mu} \pi_{\nu)}, \quad (30)$$

$$f_{\mu\nu}^{(2)} = \bar{\nabla}_\mu \pi_\rho \bar{\nabla}_\nu \pi^\rho - \bar{R}_{\mu\rho\nu\sigma} \pi^\rho \pi^\sigma, \quad (31)$$

$$f_{\mu\nu}^{(3)} = \frac{1}{3} \bar{\nabla}_\lambda \bar{R}_{\mu\rho\nu\sigma} \pi^\lambda \pi^\rho \pi^\sigma + \frac{2}{3} \bar{R}_{\mu\rho\lambda\sigma} \bar{\nabla}_\nu \pi^\lambda \pi^\rho \pi^\sigma + \frac{2}{3} \bar{R}_{\nu\rho\lambda\sigma} \bar{\nabla}_\mu \pi^\lambda \pi^\rho \pi^\sigma. \quad (32)$$

It is not surprising that the acceleration a^μ drops out in the above expression. In fact, Eqs. (29)–(32) can also be derived by replacing ξ^μ by π^μ in Eq. (25) and taking into account that $a^\mu \equiv 0$ (since π^μ is the tangent vector of geodesics) when evaluating the Lie derivatives. This is also one of the advantages of defining the Goldstone modes π^μ using the standard Riemann normal coordinates.

Having the above results in hand, we are now ready to expand the covariant metric perturbation $H_{\mu\nu} \equiv g_{\mu\nu} - f_{\mu\nu}$ as

$$H_{\mu\nu} = H_{\mu\nu}^{(1)} + H_{\mu\nu}^{(2)} + H_{\mu\nu}^{(3)} + \mathcal{O}(\pi^4), \quad (33)$$

with

$$H_{\mu\nu}^{(1)} = h_{\mu\nu} + 2\bar{\nabla}_{(\mu} \pi_{\nu)}, \quad (34)$$

$$H_{\mu\nu}^{(2)} = -\bar{\nabla}_\mu \pi_\rho \bar{\nabla}_\nu \pi^\rho + \bar{R}_{\mu\rho\nu\sigma} \pi^\rho \pi^\sigma, \quad (35)$$

$$H_{\mu\nu}^{(3)} = -\frac{1}{3} \bar{\nabla}_\lambda \bar{R}_{\mu\rho\nu\sigma} \pi^\lambda \pi^\rho \pi^\sigma - \frac{2}{3} \bar{R}_{\mu\rho\lambda\sigma} \bar{\nabla}_\nu \pi^\lambda \pi^\rho \pi^\sigma - \frac{2}{3} \bar{R}_{\nu\rho\lambda\sigma} \bar{\nabla}_\mu \pi^\lambda \pi^\rho \pi^\sigma. \quad (36)$$

In order to expand the matrix \mathcal{K}^μ_ν in terms of π^μ , first we note, by definition (7), that

$$\begin{aligned} \mathcal{K}^\mu_\nu &\equiv \delta^\mu_\nu - \sqrt{g^{\mu\rho} f_{\rho\nu}} \\ &= \delta^\mu_\nu - \sqrt{\delta^\mu_\nu - (g^{-1}H)^\mu_\nu} \\ &= \frac{1}{2} (g^{-1}H)^\mu_\nu + \frac{1}{8} (g^{-1}H)^2{}^\mu_\nu + \frac{1}{16} (g^{-1}H)^3{}^\mu_\nu + \dots, \end{aligned} \quad (37)$$

² The same expression is used in cosmological perturbations of massive gravity around an open-Friedmann-Robertson-Walker fiducial metric [26]. Similar higher order Lie derivatives also naturally arise when constructing higher-order gauge-invariant cosmological perturbations, see e.g. [27].

with $(g^{-1}H)^\mu_\nu \equiv g^{\mu\rho}H_{\rho\nu}$ etc. When $g_{\mu\nu}$ and $f_{\mu\nu}$ are expanded around the same background (as we will do in this paper), $H_{\mu\nu}$ is a perturbative quantity and hence we can calculate the action order by order through this equation.³ Plugging Eqs. (34)–(36) into Eq. (37) and using $g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu} + h^\mu_\rho h^{\rho\nu} + \dots$, \mathcal{K}^μ_ν can be expanded as

$$\mathcal{K}^\mu_\nu = \mathcal{K}^{(1)\mu}_\nu + \mathcal{K}^{(2)\mu}_\nu + \mathcal{K}^{(3)\mu}_\nu + \dots, \quad (38)$$

with

$$\mathcal{K}^{(1)\mu}_\nu = \frac{1}{2}h^\mu_\nu + \Pi^\mu_\nu, \quad (39)$$

$$\mathcal{K}^{(2)\mu}_\nu = -\frac{3}{8}h^{\mu\rho}h_{\rho\nu} - \frac{3}{4}h^{\mu\rho}\Pi_{\rho\nu} + \frac{1}{4}\Pi^{\mu\rho}h_{\rho\nu} - \frac{1}{2}\bar{\nabla}^\mu\pi^\rho\bar{\nabla}_\nu\pi_\rho + \frac{1}{2}\Pi^{\mu\rho}\Pi_{\rho\nu} + \frac{1}{2}\bar{R}^\mu_{\rho\nu\sigma}\pi^\rho\pi^\sigma, \quad (40)$$

$$\begin{aligned} \mathcal{K}^{(3)\mu}_\nu = & \frac{5}{16}h^{\mu\rho}h_{\rho\sigma}h^\sigma_\nu + \frac{5}{8}h^{\mu\rho}h_{\rho\sigma}\Pi^\sigma_\nu - \frac{1}{8}h^{\mu\rho}\Pi_{\rho\sigma}h^\sigma_\nu - \frac{1}{8}\Pi^{\mu\rho}h_{\rho\sigma}h^\sigma_\nu \\ & - \frac{1}{4}h^{\mu\rho}\Pi_{\rho\sigma}\Pi^\sigma_\nu - \frac{1}{4}\Pi^{\mu\rho}h_{\rho\sigma}\Pi^\sigma_\nu + \frac{1}{4}\Pi^{\mu\rho}\Pi_{\rho\sigma}h^\sigma_\nu + \frac{3}{8}h^{\mu\rho}\bar{\nabla}_\rho\pi_\sigma\bar{\nabla}_\nu\pi^\sigma - \frac{1}{8}\bar{\nabla}^\mu\pi_\rho\bar{\nabla}^\sigma\pi^\rho h_{\sigma\nu} \\ & - \frac{3}{8}h^{\mu\lambda}\bar{R}_{\lambda\rho\nu\sigma}\pi^\rho\pi^\sigma + \frac{1}{8}\bar{R}^\mu_{\rho\lambda\sigma}\pi^\rho\pi^\sigma h^\lambda_\nu \\ & + \frac{1}{2}\Pi^{\mu\rho}\Pi_{\rho\sigma}\Pi^\sigma_\nu - \frac{1}{4}(\Pi^{\mu\rho}\bar{\nabla}_\rho\pi_\sigma\bar{\nabla}_\nu\pi^\sigma + \Pi_{\nu}{}^\rho\bar{\nabla}_\rho\pi_\sigma\bar{\nabla}^\mu\pi^\sigma) - \frac{1}{6}\bar{\nabla}_\lambda\bar{R}^\mu_{\rho\nu\sigma}\pi^\lambda\pi^\rho\pi^\sigma \\ & - \frac{1}{3}(\bar{R}^\mu_{\rho\lambda\sigma}\bar{\nabla}_\nu\pi^\lambda + \bar{R}_{\nu\rho\lambda\sigma}\bar{\nabla}^\mu\pi^\lambda) + \frac{1}{4}(\Pi^{\mu\lambda}\bar{R}_{\lambda\rho\nu\sigma}\pi^\rho\pi^\sigma + \Pi_{\nu}{}^\lambda\bar{R}_{\lambda\rho}{}^\mu{}_\sigma\pi^\rho\pi^\sigma), \end{aligned} \quad (41)$$

where $\Pi_{\mu\nu}$ is the symmetric part of $\bar{\nabla}_\mu\pi_\nu$:

$$\Pi_{\mu\nu} \equiv \frac{1}{2}(\bar{\nabla}_\mu\pi_\nu + \bar{\nabla}_\nu\pi_\mu). \quad (42)$$

Finally, putting everything together and using the expansion for the determinant, $\sqrt{-g} = \sqrt{-\bar{g}}\{1 + h/2 + (h^2 - 2h^{\mu\nu}h_{\mu\nu})/8 + \dots\}$, we are able to expand the full dRGT mass terms (1) in terms of the metric perturbation $h_{\mu\nu}$ and the Goldstone modes π^μ order by order. At second order, we have

$$(\sqrt{-g}\mathcal{L}^{\text{dRGT}})^{(2)} = \sqrt{-\bar{g}}(\mathcal{L}_{h^2} + \mathcal{L}_{h\pi} + \mathcal{L}_{\pi^2}), \quad (43)$$

with

$$\mathcal{L}_{h^2} = \frac{1}{4}(h^2 - h_{\mu\nu}h^{\mu\nu}), \quad (44)$$

$$\mathcal{L}_{h\pi} = h^{\mu\nu}(\Pi\bar{g}_{\mu\nu} - \Pi_{\mu\nu}), \quad (45)$$

$$\mathcal{L}_{\pi^2} = -F_{\mu\nu}F^{\mu\nu} + \bar{R}_{\mu\nu}\pi^\mu\pi^\nu, \quad (46)$$

where $F_{\mu\nu}$ is the antisymmetric part of $\bar{\nabla}_\mu\pi_\nu$,

$$F_{\mu\nu} \equiv \frac{1}{2}(\bar{\nabla}_\mu\pi_\nu - \bar{\nabla}_\nu\pi_\mu), \quad (47)$$

and we omitted a total divergence in Eq. (46). Similarly, the Lagrangian at cubic order is

$$\{\sqrt{-g}\mathcal{L}^{\text{dRGT}}\}^{(3)} = \sqrt{-\bar{g}}(\mathcal{L}_{h^3} + \mathcal{L}_{h^2\pi} + \mathcal{L}_{h\pi^2} + \mathcal{L}_{\pi^3}), \quad (48)$$

where

$$\mathcal{L}_{h^3} = \frac{1}{8}[(1 + \alpha_3)h^3 - (4 + 3\alpha_3)hh_{\mu\nu}h^{\mu\nu} + (3 + 2\alpha_3)h^\mu_\nu h^\nu_\rho h^\rho_\mu], \quad (49)$$

$$\mathcal{L}_{h^2\pi} = \frac{1}{4}[(5 + 6\alpha_3)h^\mu_\nu h^\nu_\rho \Pi^\rho_\mu - 3(1 + \alpha_3)h_{\mu\nu}h^{\mu\nu}\Pi - (2 + 3\alpha_3)h(2h^{\nu\rho}\Pi_{\nu\rho} - h\Pi)], \quad (50)$$

$$\begin{aligned} \mathcal{L}_{h\pi^2} = & \frac{1}{2}(1 + 3\alpha_3)[2h^{\mu\nu}(\Pi^\rho_\mu\Pi_{\nu\rho} - \Pi\Pi_{\mu\nu}) + h(\Pi^2 - \Pi_{\nu\rho}\Pi^{\nu\rho})] \\ & + \frac{1}{2}(F^\rho_\mu h_{\nu\rho} - F_{\mu\nu}h + 2\Pi_{\nu\rho}h^\rho_\mu)F^{\mu\nu} + \frac{1}{2}(h\bar{R}_{\mu\nu} - h^{\rho\sigma}\bar{R}_{\mu\rho\nu\sigma})\pi^\mu\pi^\nu, \end{aligned} \quad (51)$$

³ Note that the square root matrix can be also expanded easily in the proportional background case: $\bar{g}_{\mu\nu} = C^2 g_{\mu\nu}^{(0)}$. Our following analysis can be easily extended to this case. The only difference is existence of the overall factor C and $\mathcal{K}^{(0)\mu}_\nu = (1 - C)\delta^\mu_\nu$.

and

$$\mathcal{L}_{\pi^3} = (F_\mu^\rho \Pi_{\nu\rho} - F_{\mu\nu} \Pi) F^{\mu\nu} + (\bar{R}_{\mu\nu} \Pi - \bar{R}_{\mu\rho\nu\sigma} \Pi^{\rho\sigma}) \pi^\mu \pi^\nu + \alpha_3 (\Pi^3 - 3\Pi\Pi_{\nu\rho} \Pi^{\nu\rho} + 2\Pi_{\nu\rho} \Pi_\mu^\rho \Pi^{\mu\nu}). \quad (52)$$

At quartic order we have

$$\{\sqrt{-g}\mathcal{L}^{\text{dRGT}}\}^{(4)} = \sqrt{-g} (\mathcal{L}_{h^4} + \mathcal{L}_{h^3\pi} + \mathcal{L}_{h^2\pi^2} + \mathcal{L}_{h\pi^3} + \mathcal{L}_{\pi^4}), \quad (53)$$

where

$$\begin{aligned} \mathcal{L}_{h^4} = & \frac{1}{32} (1 + 2\alpha_3 + 2\alpha_4) h^4 - \frac{3}{32} (3 + 5\alpha_3 + 4\alpha_4) h^2 h_{\mu\nu} h^{\mu\nu} + \frac{1}{16} (8 + 11\alpha_3 + 8\alpha_4) h h_\nu^\mu h_\rho^\nu h_\mu^\rho \\ & + \frac{1}{64} (13 + 18\alpha_3 + 12\alpha_4) (h_{\mu\nu} h^{\mu\nu})^2 - \frac{1}{64} (29 + 36\alpha_3 + 24\alpha_4) h_\nu^\mu h_\rho^\nu h_\sigma^\rho h_\mu^\sigma, \end{aligned} \quad (54)$$

$$\begin{aligned} \mathcal{L}_{h^3\pi} = & -\frac{1}{8} (11 + 24\alpha_3 + 24\alpha_4) h_\mu^\rho h^{\mu\nu} h_\nu^\sigma \Pi_{\rho\sigma} + \frac{1}{8} (8 + 21\alpha_3 + 24\alpha_4) h h_\nu^\sigma h^{\nu\rho} \Pi_{\rho\sigma} \\ & + \frac{1}{8} (5 + 12\alpha_3 + 12\alpha_4) h_{\mu\nu} h^{\mu\nu} (h^{\rho\sigma} \Pi_{\rho\sigma} - h\Pi) + \frac{1}{8} (5 + 9\alpha_3 + 8\alpha_4) h_\mu^\rho h^{\mu\nu} h_{\nu\rho} \Pi \\ & + \frac{1}{8} (1 + 3\alpha_3 + 4\alpha_4) h^2 (h\Pi - 3h^{\mu\nu} \Pi_{\mu\nu}), \end{aligned} \quad (55)$$

$$\begin{aligned} \mathcal{L}_{h^2\pi^2} = & -\frac{1}{8} (5 + 6\alpha_3) (F_\lambda^\nu F^{\lambda\mu} h_\mu^\rho h_{\nu\rho} + 2F^{\lambda\mu} h_\lambda^\nu h_\nu^\rho \Pi_{\mu\rho} - h_\nu^\sigma h^{\nu\rho} \bar{R}_{\lambda\rho\mu\sigma} \pi^\lambda \pi^\mu) \\ & + \frac{3}{8} (1 + \alpha_3) h_{\nu\rho} h^{\nu\rho} (F_{\lambda\mu} F^{\lambda\mu} - \bar{R}_{\lambda\mu} \pi^\lambda \pi^\mu) + \frac{1}{8} (2 + 3\alpha_3) (2F_\lambda^\nu F^{\lambda\mu} h_{\mu\nu} h - F_{\lambda\mu} F^{\lambda\mu} h^2) \\ & + \frac{1}{8} (2 + 3\alpha_3) (4F^{\lambda\mu} h_\lambda^\nu h \Pi_{\mu\nu} + \bar{R}_{\mu\rho\nu\sigma} h (h\bar{g}^{\rho\sigma} - 2h^{\rho\sigma}) \pi^\mu \pi^\nu) \\ & + \frac{1}{8} (1 + 6\alpha_3 + 12\alpha_4) [2(h^{\mu\nu} \Pi_{\mu\nu})^2 + 4h h^{\mu\nu} (\Pi_\mu^\rho \Pi_{\nu\rho} - \Pi_{\mu\nu} \Pi) + h^2 (\Pi^2 - \Pi_{\mu\nu} \Pi^{\mu\nu})] \\ & + \frac{1}{8} (2 + 9\alpha_3 + 12\alpha_4) h_{\mu\nu} h^{\mu\nu} (\Pi_{\rho\sigma} \Pi^{\rho\sigma} - \Pi^2) - \frac{1}{8} (7 + 30\alpha_3 + 48\alpha_4) h_\lambda^\nu h^{\lambda\mu} \Pi_\mu^\rho \Pi_{\nu\rho} \\ & + \frac{3}{4} (1 + 5\alpha_3 + 8\alpha_4) h_\lambda^\nu h^{\lambda\mu} \Pi_{\mu\nu} \Pi - \frac{1}{8} (1 + 12\alpha_3 + 24\alpha_4) h^{\lambda\mu} h^{\nu\rho} \Pi_{\lambda\nu} \Pi_{\mu\rho}, \end{aligned} \quad (56)$$

$$\begin{aligned} \mathcal{L}_{h\pi^3} = & \frac{1}{2} [3\alpha_3 F_\lambda^\nu h + 2(1 + 3\alpha_3) h_\lambda^\nu \Pi] F^{\lambda\mu} \Pi_{\mu\nu} - \frac{1}{2} (1 + 6\alpha_3) (F_\lambda^\nu F^{\lambda\mu} h_\mu^\rho \Pi_{\nu\rho} + F^{\lambda\mu} h_\lambda^\nu \Pi_\mu^\rho \Pi_{\nu\rho}) \\ & + \frac{1}{2} (1 + 3\alpha_3) [F_{\lambda\mu} F^{\lambda\mu} (h^{\nu\rho} \Pi_{\nu\rho} - h\Pi) + F_\lambda^\nu F^{\lambda\mu} h_{\mu\nu} \Pi + (\bar{g}_{\nu\rho} \bar{R}_{\lambda\mu} \bar{g}_{\alpha\beta} - \bar{R}_{\lambda\mu} \bar{g}_{\nu\alpha} \bar{g}_{\rho\beta} - \bar{R}_{\lambda\nu\mu\rho} \bar{g}_{\alpha\beta}) h^{\nu\rho} \Pi^{\alpha\beta} \pi^\lambda \pi^\mu] \\ & + \frac{1}{2} (\alpha_3 + 4\alpha_4) [h^{\lambda\mu} (3\Pi_{\lambda\mu} (\Pi_{\nu\rho} \Pi^{\nu\rho} - \Pi^2) + 6\Pi_\lambda^\nu (\Pi_{\mu\nu} \Pi - \Pi_\mu^\rho \Pi_{\nu\rho})) + h (\Pi^3 - 3\Pi\Pi_{\nu\rho} \Pi^{\nu\rho} + 2\Pi_\mu^\rho \Pi^{\mu\nu} \Pi_{\nu\rho})] \\ & + \frac{1}{6} \pi^\lambda \pi^\mu [(6F^{\nu\rho} h_\nu^\sigma + (7 + 18\alpha_3) h^{\nu\rho} \Pi_\nu^\sigma - (4 + 9\alpha_3) h \Pi^{\rho\sigma}) \bar{R}_{\lambda\rho\mu\sigma} - (h\bar{\nabla}_\nu \bar{R}_{\lambda\mu} - h^{\rho\sigma} \bar{\nabla}_\nu \bar{R}_{\lambda\rho\mu\sigma}) \pi^\nu], \end{aligned} \quad (57)$$

and

$$\begin{aligned} \mathcal{L}_{\pi^4} = & \frac{1}{4} F_\lambda^\nu F^{\lambda\mu} [2(1 - 6\alpha_3) \Pi_\mu^\rho \Pi_{\nu\rho} - 4(1 - 3\alpha_3) \Pi_{\mu\nu} \Pi - F_\mu^\rho F_{\nu\rho}] + \frac{1}{4} (F_{\lambda\mu} F^{\lambda\mu})^2 + \frac{1}{2} F^{\lambda\mu} F^{\nu\rho} \Pi_{\lambda\nu} \Pi_{\mu\rho} \\ & + \alpha_4 (\Pi^4 + 8\Pi\Pi_\mu^\rho \Pi^{\mu\nu} \Pi_{\nu\rho} + 3(\Pi_{\lambda\mu} \Pi^{\lambda\mu})^2 - 6\Pi^2 \Pi_{\nu\rho} \Pi^{\nu\rho} - 6\Pi_\lambda^\nu \Pi^{\lambda\mu} \Pi_\mu^\rho \Pi_{\nu\rho}) \\ & - \frac{3}{2} \alpha_3 (F_{\lambda\mu} F^{\lambda\mu} - \bar{R}_{\lambda\mu} \pi^\lambda \pi^\mu) (\Pi^2 - \Pi_{\nu\rho} \Pi^{\nu\rho}) + \frac{1}{6} \bar{R}_{\lambda\rho\mu\sigma} (3F_\nu^\sigma F^{\nu\rho} + 2F^{\nu\rho} \Pi_\nu^\sigma - 3F_{\nu\alpha} F^{\nu\alpha} \bar{g}^{\rho\sigma}) \pi^\lambda \pi^\mu \\ & + \frac{1}{3} (1 + 9\alpha_3) \bar{R}_{\lambda\rho\mu\sigma} (\Pi_\nu^\sigma \Pi^{\nu\rho} - \Pi\Pi^{\rho\sigma}) \pi^\lambda \pi^\mu + \frac{1}{4} (\bar{R}_{\lambda\mu} \bar{R}_{\nu\rho} - \bar{R}_\lambda^\sigma{}_\mu{}^\tau \bar{R}_{\nu\sigma\rho\tau}) \pi^\lambda \pi^\mu \pi^\nu \pi^\rho \\ & - \frac{1}{3} \bar{\nabla}_\nu (\bar{g}_{\rho\sigma} \bar{R}_{\lambda\mu} - \bar{R}_{\lambda\rho\mu\sigma}) \Pi^{\rho\sigma} \pi^\lambda \pi^\mu \pi^\nu. \end{aligned} \quad (58)$$

The above expressions are very cumbersome. In the next section, we will show that it is possible to introduce a generalized Λ_3 -decoupling limit as in the case of the flat fiducial metric. All terms with cutoff scales lower than Λ_3 drop out and the resulting terms represent a healthy theory describing various modes propagating on a curved background.

IV. DECOUPLING LIMIT AND THE HELICITY-0 MODE

A. Scales

The covariant approach employed in the previous sections enables us to identify the propagating degrees of freedom correctly. We may split π_μ into transverse and longitudinal modes as in the case of the flat fiducial metric:

$$\pi_\mu = A_\mu + \bar{\nabla}_\mu \pi = \frac{1}{M_{\text{pl}} m} \hat{A}_\mu + \frac{1}{M_{\text{pl}} m^2} \bar{\nabla}_\mu \hat{\pi}, \quad (59)$$

where \hat{A}_μ and $\hat{\pi}$ are normalized and are identified as the helicity-1 and helicity-0 modes, respectively. Similarly, we define the normalized $\hat{h}_{\mu\nu}$ as

$$\hat{h}_{\mu\nu} = M_{\text{pl}} h_{\mu\nu}. \quad (60)$$

The Stückelberg expansion yields a whole hierarchy of interaction terms of $\hat{h}_{\mu\nu}$, \hat{A}_μ , and $\hat{\pi}$ with various energy scales. Note that while $h_{\mu\nu}$ without derivatives appear in the expansion, π_μ without derivatives does not in the case of the flat fiducial metric. This point should be contrasted with the curved case, as π_μ now may appear without derivatives due to the presence of the curvature tensor and its derivatives, which come from the commutation of the covariant derivatives. Thus, a general interaction term takes the following prototype

$$M_{\text{pl}}^2 m^2 (\bar{\nabla}^d \bar{R}^r) h^{n_h} A^a (\bar{\nabla} A)^{n_A - a} (\bar{\nabla} \pi)^{2r + d - a} (\bar{\nabla}^2 \pi)^{n_\pi - 2r - d + a}, \quad (61)$$

where n_h , n_A , and n_π are the numbers of the corresponding fields, r is the power of curvature terms, and d is the number of derivatives acting on the curvature. All the powers in Eq. (61) must be non-negative integers so that, especially,

$$0 \leq a \leq n_A, \quad 0 \leq 2r + d - a \leq n_\pi. \quad (62)$$

In terms of the normalized variables, Eq. (61) can be written as

$$\frac{1}{\Lambda_\lambda^p} \left(\frac{\bar{\nabla}^d \bar{R}^r}{m^{2r}} \right) \times \hat{h}^{n_h} \hat{A}^a (\bar{\nabla} \hat{A})^{n_A - a} (\bar{\nabla} \hat{\pi})^{2r + d - a} (\bar{\nabla}^2 \hat{\pi})^{n_\pi + a - 2r - d}, \quad (63)$$

where Λ_λ is defined as usual as $\Lambda_\lambda \equiv (M_{\text{pl}} m^{\lambda-1})^{1/\lambda}$ with

$$p = n_h + 2n_A + 3n_\pi - 4 - 2r, \quad \lambda = \frac{n_h + 2n_A + 3n_\pi - 4 - 2r}{n_h + n_A + n_\pi - 2}. \quad (64)$$

Note that in Eq. (63) we deliberately separate the dimensionless (nondynamical) factor \bar{R}^r/m^{2r} for later convenience. At this point, it is clear that the only difference from the case of the flat fiducial metric is the presence of the curvature terms in Eq. (63), which effectively change the cutoff scales. Equations (63) and (64) generalize the expressions for the case of the flat fiducial metric ($r = d = a = 0$) to the curved case.

In Appendix B, we list all possible interaction terms with corresponding cutoff scales, up to fourth order in powers of fields. In general, there are two types of terms suppressed by scales lower than Λ_3 :

$$\frac{1}{\Lambda_{(3n_\pi-4)/(n_\pi-2)}} (\bar{\nabla}^2 \hat{\pi})^{n_\pi}, \quad \frac{1}{\Lambda_{(3n_\pi-2)/(n_\pi-1)}} (\bar{\nabla} \hat{A}) (\bar{\nabla}^2 \hat{\pi})^{n_\pi}, \quad (65)$$

which are exactly the same as the case of the flat fiducial metric. On the other hand, terms suppressed by Λ_3 are:

$$\frac{1}{\Lambda_3^{3n_\pi-3}} \times \hat{h} (\bar{\nabla}^2 \hat{\pi})^{n_\pi}, \quad \frac{1}{\Lambda_3^{3n_\pi}} (\bar{\nabla} \hat{A})^2 (\bar{\nabla}^2 \hat{\pi})^{n_\pi}, \quad \frac{1}{\Lambda_3^{3n_\pi-6}} \left(\frac{\bar{\nabla}^d \bar{R}}{m^2} \right) \times (\bar{\nabla} \hat{\pi})^{2+d} (\bar{\nabla}^2 \hat{\pi})^{n_\pi-2-d}, \quad (66)$$

where the last type of terms arises due to the presence of the curvature of the fiducial metric. All the other terms are suppressed by scales higher than Λ_3 .

B. Extended Λ_3 -decoupling limit

In the case of the flat fiducial metric, the “ Λ_λ -decoupling limit” is taken as

$$m \rightarrow 0, \quad M_{\text{pl}} \rightarrow \infty, \quad \Lambda_\lambda \equiv (M_{\text{pl}} m^{\lambda-1})^{1/\lambda} = \text{const.}$$

For example, the cutoff scale of nonlinear dRGT massive gravity is Λ_3 . As we shall see, this holds also for the case of a curved fiducial metric. Thus, we will take an “extended” Λ_3 decoupling limit as

$$m \rightarrow 0, \quad M_{\text{pl}} \rightarrow \infty, \quad \Lambda_3 \equiv (M_{\text{pl}} m^2)^{1/3} = \text{const.}, \quad \frac{\bar{R}_{\mu\nu\rho\sigma}}{m^2} \rightarrow \text{finite.} \quad (67)$$

Note that the interactions between \hat{A}_μ and $\hat{\pi}$ start at cubic order. Therefore, similarly to the analysis of the flat fiducial metric, we consistently set $\hat{A}_\mu = 0$ and concentrate on $\hat{h}_{\mu\nu}$ and $\hat{\pi}$ in the following.

After taking this extended Λ_3 decoupling limit, the surviving interaction terms up to fourth order in fields are given by

$$\begin{aligned} \sqrt{-g} M_{\text{pl}}^2 m^2 \mathcal{L}^{\text{dRGT}} \xrightarrow{\text{D.L.}} \sqrt{-\bar{g}} \left[\hat{h}^{\mu\nu} \left(X_{\mu\nu}^{(1)}(\hat{\pi}) + \frac{1+3\alpha_3}{2\Lambda_3^3} X_{\mu\nu}^{(2)}(\hat{\pi}) + \frac{\alpha_3+4\alpha_4}{2\Lambda_3^6} X_{\mu\nu}^{(3)}(\hat{\pi}) \right) \right. \\ \left. + \mathcal{L}_{\hat{\pi}^2} + \mathcal{L}_{\hat{\pi}^3} + \mathcal{L}_{\hat{\pi}^4} \right], \end{aligned} \quad (68)$$

where

$$X_{\mu\nu}^{(1)}(\hat{\pi}) = \bar{g}_{\mu\nu} \bar{\square} \hat{\pi} - \bar{\nabla}_\mu \bar{\nabla}_\nu \hat{\pi}, \quad (69)$$

$$X_{\mu\nu}^{(2)}(\hat{\pi}) = \bar{g}_{\mu\nu} \left((\bar{\square} \hat{\pi})^2 - \bar{\nabla}_\rho \bar{\nabla}_\sigma \hat{\pi} \bar{\nabla}^\rho \bar{\nabla}^\sigma \hat{\pi} \right) + 2 \left(\bar{\nabla}_\mu \bar{\nabla}_\rho \hat{\pi} \bar{\nabla}^\rho \bar{\nabla}_\nu \hat{\pi} - \bar{\square} \hat{\pi} \bar{\nabla}_\mu \bar{\nabla}_\nu \hat{\pi} \right), \quad (70)$$

$$\begin{aligned} X_{\mu\nu}^{(3)}(\hat{\pi}) \equiv \bar{g}_{\mu\nu} \left((\bar{\square} \hat{\pi})^3 - 3 \bar{\square} \hat{\pi} \bar{\nabla}_\rho \bar{\nabla}_\sigma \hat{\pi} \bar{\nabla}^\rho \bar{\nabla}^\sigma \hat{\pi} + 2 \bar{\nabla}^\rho \bar{\nabla}_\sigma \hat{\pi} \bar{\nabla}^\sigma \bar{\nabla}_\lambda \hat{\pi} \bar{\nabla}^\lambda \bar{\nabla}_\rho \hat{\pi} \right) \\ + 3 \bar{\nabla}_\mu \bar{\nabla}_\nu \hat{\pi} \left(\bar{\nabla}_\rho \bar{\nabla}_\sigma \hat{\pi} \bar{\nabla}^\rho \bar{\nabla}^\sigma \hat{\pi} - (\bar{\square} \hat{\pi})^2 \right) + 6 \bar{\nabla}^\rho \bar{\nabla}_\mu \hat{\pi} \left(\bar{\nabla}_\nu \bar{\nabla}_\rho \hat{\pi} \bar{\square} \hat{\pi} - \bar{\nabla}_\nu \bar{\nabla}^\sigma \hat{\pi} \bar{\nabla}_\rho \bar{\nabla}_\sigma \hat{\pi} \right), \end{aligned} \quad (71)$$

and

$$\mathcal{L}_{\hat{\pi}^2} \simeq \frac{\bar{R}_{\mu\nu}}{m^2} \bar{\nabla}^\mu \hat{\pi} \bar{\nabla}^\nu \hat{\pi}, \quad (72)$$

$$\mathcal{L}_{\hat{\pi}^3} \simeq \frac{1}{\Lambda_3^3} \mathcal{A}_{\mu\nu\rho\sigma} \bar{\nabla}^\mu \hat{\pi} \bar{\nabla}^\nu \hat{\pi} \bar{\nabla}^\rho \bar{\nabla}^\sigma \hat{\pi}, \quad (73)$$

$$\mathcal{L}_{\hat{\pi}^4} \simeq \frac{1}{\Lambda_3^6} \left(\mathcal{B}_{\mu\nu\rho\sigma\rho'\sigma'} \bar{\nabla}^{\rho'} \bar{\nabla}^{\sigma'} \hat{\pi} - \frac{1}{3} \mathcal{C}_{\lambda\mu\nu\rho\sigma} \bar{\nabla}^\lambda \hat{\pi} \right) \bar{\nabla}^\mu \hat{\pi} \bar{\nabla}^\nu \hat{\pi} \bar{\nabla}^\rho \bar{\nabla}^\sigma \hat{\pi}, \quad (74)$$

with⁴

$$\mathcal{A}_{\mu\nu\rho\sigma} \equiv \frac{1}{m^2} \left[(1+2\alpha_3) (\bar{R}_{\mu\nu} \bar{g}_{\rho\sigma} + \bar{R}_{\rho(\mu\nu)\sigma}) - \alpha_3 (\bar{g}_{\rho(\mu} \bar{R}_{\nu)\sigma} + \bar{g}_{\sigma(\mu} \bar{R}_{\nu)\rho}) \right], \quad (75)$$

$$\begin{aligned} \mathcal{B}_{\mu\nu\rho\sigma\rho'\sigma'} \equiv \frac{1}{m^2} \left[\frac{3}{2} (\alpha_3 + 2\alpha_4) \bar{R}_{\mu\nu} (2\bar{g}_{\rho[\sigma} \bar{g}_{\sigma']\rho'}) + 12\alpha_4 \bar{R}_{\mu[\rho} \bar{g}_{\rho']\sigma} \bar{g}_{\sigma']\nu} \right. \\ \left. - \frac{1}{3} (1+9\alpha_3+18\alpha_4) (\bar{R}_{\mu\rho\nu[\sigma} \bar{g}_{\sigma']\rho'} - \bar{R}_{\mu\rho'\nu[\sigma} \bar{g}_{\sigma']\rho}) - 6\alpha_4 \bar{g}_{\mu[\rho} \bar{R}_{\rho']\nu\sigma\sigma'} \right], \end{aligned} \quad (76)$$

$$\mathcal{C}_{\lambda\mu\nu\rho\sigma} \equiv \frac{1}{m^2} \left[\bar{g}_{\rho\sigma} \bar{\nabla}_{(\lambda} \bar{R}_{\mu\nu)} + \frac{1}{3} (\bar{\nabla}_\lambda \bar{R}_{\mu(\rho\sigma)\nu} + \bar{\nabla}_\mu \bar{R}_{\lambda(\rho\sigma)\nu} + \bar{\nabla}_\nu \bar{R}_{\lambda(\rho\sigma)\mu}) \right]. \quad (77)$$

It is not surprising that the self-interactions of $\hat{\pi}$ are all proportional to the curvature of the fiducial metric, which exactly vanish in the flat limit. Note that in deriving Eqs. (72), (73), and (74), we employed several integrations by parts, see Appendix D for details. For later convenience, note also that $\mathcal{B}_{\mu\nu\rho\sigma\rho'\sigma'}$ has the following antisymmetries:

$$\mathcal{B}_{\mu\nu\rho\sigma\rho'\sigma'} = -\mathcal{B}_{\mu\nu\rho'\sigma\rho\sigma'} = -\mathcal{B}_{\mu\nu\rho\sigma\rho'\sigma'}. \quad (78)$$

⁴ Here the (anti)symmetrization is normalized, e.g., $A_{(\mu} B_{\nu)} = \frac{1}{2} (A_\mu B_\nu + A_\nu B_\mu)$ etc.

C. Unmixing $\hat{h}_{\mu\nu}$ and $\hat{\pi}$

Unlike the case of the flat fiducial metric, here $\hat{\pi}$ acquires a quadratic kinetic term (72) automatically, due to the nonvanishing curvature of the fiducial metric. Nevertheless, it is interesting to perform the field redefinition

$$\hat{h}_{\mu\nu} \rightarrow \hat{h}_{\mu\nu} + \hat{\pi} \bar{g}_{\mu\nu} - \frac{1 + 3\alpha_3}{\Lambda_3^3} \bar{\nabla}_\mu \hat{\pi} \bar{\nabla}_\nu \hat{\pi}, \quad (79)$$

under which $\hat{h}^{\mu\nu} X_{\mu\nu}^{(1)}$ and $\hat{h}^{\mu\nu} X_{\mu\nu}^{(2)}$ get unmixed as in the case of the flat fiducial metric. To this end, we first expand the Einstein-Hilbert Lagrangian around a general background up to quadratic order,

$$\left\{ \frac{1}{2} M_{\text{pl}}^2 \sqrt{-g} R \right\}_2 \simeq -\frac{1}{4} \sqrt{-\bar{g}} \hat{h}_{\mu\nu} \mathcal{E}^{\mu\nu, \rho\sigma} \hat{h}_{\rho\sigma}, \quad (80)$$

where the ‘‘Lichnerowicz operator’’ is defined by

$$\begin{aligned} \mathcal{E}_{\mu\nu, \rho\sigma} \hat{h}^{\rho\sigma} \equiv & -\frac{1}{2} \bar{\square} \hat{h}_{\mu\nu} - \frac{1}{2} \bar{\nabla}_\mu \bar{\nabla}_\nu \hat{h} + \frac{1}{2} \bar{g}_{\mu\nu} \left(\bar{\square} \hat{h} - \bar{\nabla}_\rho \bar{\nabla}_\sigma \hat{h}^{\rho\sigma} \right) + \bar{\nabla}_\rho \bar{\nabla}_{(\mu} \hat{h}_{\nu)}^\rho \\ & - 2 \left(\hat{h}_{(\mu}^\rho \bar{R}_{\nu)\rho} - \frac{1}{2} \hat{h} \bar{R}_{\mu\nu} \right) - \frac{1}{4} \left(\bar{g}_{\mu\nu} \hat{h} - 2 \hat{h}_{\mu\nu} \right) \bar{R}. \end{aligned} \quad (81)$$

After taking the Λ_3 decoupling limit, the terms in the second line of Eq. (81) drop out and thus would not contribute to the (partially) unmixed Lagrangian.

With some manipulations, the final Lagrangian can be written as

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4, \quad (82)$$

where the quadratic terms are

$$\mathcal{L}_2 = -\frac{1}{4} \hat{h}_{\mu\nu} \mathcal{E}^{\mu\nu, \rho\sigma} \hat{h}_{\rho\sigma} - \frac{1}{2} \left(\frac{3}{2} \bar{g}_{\mu\nu} - \frac{\bar{R}_{\mu\nu}}{m^2} \right) \bar{\nabla}^\mu \hat{\pi} \bar{\nabla}^\nu \hat{\pi}. \quad (83)$$

It is interesting to see that, at linear order, $\hat{\pi}$ propagates in an effective metric $(3/2)\bar{g}_{\mu\nu} - \bar{R}_{\mu\nu}/m^2$. Thus, although there are no higher derivatives so that the theory is free of any extra modes, $\hat{\pi}$ itself is a ghost in spacetime regions where

$$\bar{R}_{\mu\nu} > \frac{3}{2} m^2 \bar{g}_{\mu\nu}, \quad (84)$$

is satisfied. This generalizes the well-known ‘‘Higuchi bound’’ in the de Sitter background [29]. A critical case arises for $\bar{R}_{\mu\nu} = (3/2)m^2 \bar{g}_{\mu\nu}$, where $\hat{\pi}$ becomes nondynamical (at the linear level). This case corresponds to the case of ‘‘partially massless’’ gravity.

The cubic and quartic parts are

$$\mathcal{L}_3 = -\frac{3(1 + 3\alpha_3)}{4\Lambda_3^3} (\bar{\nabla} \hat{\pi})^2 \bar{\square} \hat{\pi} + \frac{1}{2\Lambda_3^3} \mathcal{A}_{\mu\nu\rho\sigma} \bar{\nabla}^\mu \hat{\pi} \bar{\nabla}^\nu \hat{\pi} \bar{\nabla}^\rho \bar{\nabla}^\sigma \hat{\pi}, \quad (85)$$

and

$$\begin{aligned} \mathcal{L}_4 = & -\frac{1 + 8\alpha_3 + 9\alpha_3^2 + 8\alpha_4}{4\Lambda_3^6} (\bar{\nabla} \hat{\pi})^2 \left((\bar{\square} \hat{\pi})^2 - \bar{\nabla}_\rho \bar{\nabla}_\sigma \hat{\pi} \bar{\nabla}^\rho \bar{\nabla}^\sigma \hat{\pi} \right) + \frac{1}{4\Lambda_3^6} (\alpha_3 + 4\alpha_4) \hat{h}^{\mu\nu} X_{\mu\nu}^{(3)}(\hat{\pi}) \\ & + \frac{1}{2\Lambda_3^6} \left(\mathcal{B}_{\mu\nu\rho\sigma\rho'\sigma'} \bar{\nabla}^{\rho'} \bar{\nabla}^{\sigma'} \hat{\pi} - \frac{1}{3} \mathcal{C}_{\lambda\mu\nu\rho\sigma} \bar{\nabla}^\lambda \hat{\pi} \right) \bar{\nabla}^\mu \hat{\pi} \bar{\nabla}^\nu \hat{\pi} \bar{\nabla}^\rho \bar{\nabla}^\sigma \hat{\pi}, \end{aligned} \quad (86)$$

respectively, where $\mathcal{A}_{\mu\nu\rho\sigma}$ etc. are defined in Eqs. (75)–(77).

In the case of the flat fiducial metric, a necessary condition for the absence of the BD ghost is the disappearance of self-interactions of the helicity-0 mode $\hat{\pi}$. This is because there $\hat{\pi}$ appears always with two derivatives, $\partial_\mu \partial_\nu \hat{\pi}$, and thus any self-interaction of $\hat{\pi}$ will inevitably yield higher derivatives in the equations of motion. In our case, $\hat{\pi}$ acquires self-interactions due to the presence of the curvature tensor of the fiducial metric. However, in the extended Λ_3 decoupling limit, the equation of motion for $\hat{\pi}$ remains of second order in derivatives (acting on $\hat{\pi}$). To see this, first

note that, for \mathcal{L}_3 and for the term proportional to $\mathcal{C}_{\lambda\mu\nu\rho\sigma}$ in \mathcal{L}_4 , the second derivatives of $\hat{\pi}$ appear linearly, implying that the corresponding equation of motion for $\hat{\pi}$ is of second order in derivatives. As for the first term in \mathcal{L}_4 , though it does not take the form of covariant Galileons, which must be supplemented with a curvature term $\sim \bar{R}(\bar{\nabla}\hat{\pi})^4$, it is straightforward to check that the corresponding equation of motion for $\hat{\pi}$ is of second order. The point is that the equation of motion contains derivatives of the curvature of the fiducial metric, which are definitely safe since the fiducial metric is nondynamical. This is the same for the terms proportional to $\hat{h}^{\mu\nu}X_{\mu\nu}^{(3)}(\hat{\pi})$ and $\mathcal{B}_{\mu\nu\rho\sigma\rho'\sigma'}$ due to the antisymmetries (78), see Appendix C for explicit proofs. To summarize, similarly to the case of the flat fiducial metric, in the Λ_3 decoupling limit, $\hat{\pi}$ propagates subject to a second-order equation of motion, which prevents the BD ghost.

V. CONCLUSION

In this paper, we have extended the Stückelberg analysis, which was used in the theory on the flat fiducial metric, to the theory on a general fiducial metric. First, we have given the covariant definition of the perturbation π^μ of the Stückelberg field. Using this definition, we have expanded the action in a covariant way and given the explicit expression for the action of $h_{\mu\nu}$ and π^μ up to fourth order.

As an application of this formula, we have calculated the action of the helicity-0 mode π . From the second-order action, we have obtained the ghost-free condition (84), which is the generalization of the Higuchi bound known in the de Sitter fiducial case. Contrary to the flat fiducial case, we have faced the problem in taking the Λ_3 decoupling limit in the general fiducial case. However, we have overcome this problem by extending the Λ_3 decoupling limit, in which the curvature of the fiducial metric is scaled. In this extended Λ_3 decoupling limit, the helicity-0 mode π and helicity-2 mode $h_{\mu\nu}$ are decoupled as in the flat fiducial case. (Of course, there remains the $h^{\mu\nu}X_{\mu\nu}^{(3)}$ coupling term, which exists even in the flat fiducial case.) The decoupled action is composed of the flat result and the curvature correction. The most important result is that this curvature correction does not produce any higher derivatives, leading to the second-order equation of motion. This fact offers us a different and complementary way of clarifying the reason for the absence of the BD ghost in dRGT massive gravity on a general fiducial metric. If we go into bigravity, the fiducial metric becomes dynamical as well. Then, we need to take into account the equation of motion for the fiducial (dynamical) metric. This is still an open question and left for further investigation.

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Appendix A: ANOTHER APPROACH TO EXPAND FIDUCIAL METRIC

In the main text, we regard the Stückelberg fields as diffeomorphisms of the physical spacetime. From an alternative point of view, however, ϕ^a are simply four scalar fields living on the physical spacetime, which form a four-dimensional field space at each spacetime point. It is thus interesting to see whether the relation (25) can be reproduced from this point of view.

To this end, first we introduce the metric \bar{g}_{ab} in the field space as

$$f_{\mu\nu}(\phi) = \bar{g}_{ab}(\phi) \frac{\partial\phi^a(x)}{\partial x^\mu} \frac{\partial\phi^b(x)}{\partial x^\nu}. \quad (\text{A1})$$

The one parameter family of map ϕ_λ defined in Sec. III A corresponds to curves in the field space, which can be written as $\phi^a(\lambda)$. The unitary gauge is chosen such that $\phi^a(0) = x^\mu \delta_\mu^a$. Expanding (A1) around the unitary gauge and then setting $\lambda = -1$, we have

$$f_{\mu\nu} \equiv f_{\mu\nu}(\phi)|_{\lambda=-1} = e^{-\frac{d}{d\lambda}} f_{\mu\nu}(\phi(\lambda))|_{\lambda=0}. \quad (\text{A2})$$

Since $f_{\mu\nu}$ is a scalar in field space, $d/d\lambda$ can be replaced by the Lie derivative along the curve:

$$f_{\mu\nu}(\phi) = e^{\mathcal{L}_u} f_{\mu\nu}|_{\lambda=0}, \quad (\text{A3})$$

where u^a is the tangent vector of the curve $\phi^a(\lambda)$. Moreover, (23) implies

$$\mathcal{L}_u \left(\frac{\partial \phi^a(\lambda)}{\partial x^\mu} \right) = 0. \quad (\text{A4})$$

Thus in (A3), the Lie derivative acts only on \bar{g}_{ab} , i.e.

$$f_{\mu\nu} = \left((e^{\mathcal{L}_u} \bar{g}_{ab}) \frac{\partial \phi^a(\lambda)}{\partial x^\mu} \frac{\partial \phi^b(\lambda)}{\partial x^\nu} \right) \Big|_{\lambda'=0} = (e^{\mathcal{L}_u} \bar{g}_{ab}) \Big|_{\lambda=0} \delta_\mu^a \delta_\nu^b, \quad (\text{A5})$$

which exactly coincides with (25).

Appendix B: INTERACTION TERMS AND CUTOFF SCALES

For completeness, according to (63), here we list all possible interaction terms in the Stückelberg expansion up to the fourth order in fields as well as their corresponding cutoff scales.

- $n_h + n_A + n_\pi = 2$

(n_h, n_A, n_π)	(2, 0, 0)	(0, 2, 0)	(0, 0, 2)	(1, 1, 0)	(1, 0, 1)	(0, 1, 1)
(r, d, a)	(0, 0, 0)	(0, 0, 0)	(1, 0, 2)	(0, 0, 0)	(1, 0, 0)	(0, 0, 0)
Λ_λ^p	m^{-2}	1	m^{-2}	m^2	1	m^{-1}

In this case, there are two types of terms with lowest scales:

$$\frac{1}{m^2} (\bar{\nabla}^2 \hat{\pi})^2, \quad \frac{1}{m} (\bar{\nabla} \hat{A}) (\bar{\nabla}^2 \hat{\pi}),$$

while there are three types of terms

$$\hat{h} (\bar{\nabla}^2 \hat{\pi}), \quad (\bar{\nabla} \hat{A})^2, \quad \frac{\bar{R}}{m^2} (\bar{\nabla} \hat{\pi})^2,$$

that are “scale invariant”.

- $n_h + n_A + n_\pi = 3$

(n_h, n_A, n_π)	(3, 0, 0)	(0, 3, 0)	(0, 0, 3)	(2, 1, 0)
(r, d, a)	(0, 0, 0)	(0, 0, 0)	(1, 0, 2)	(1, 1, 3)
Λ_λ^p	M_{pl}/m^2	Λ_2^2	M_{pl}/m	Λ_5^5

(n_h, n_A, n_π)	(1, 2, 0)	(2, 0, 1)	(1, 0, 2)	(0, 2, 1)
(r, d, a)	(0, 0, 0)	(1, 0, 2)	(0, 0, 0)	(0, 0, 0)
Λ_λ^p	M_{pl}	M_{pl}/m^2	M_{pl}	Λ_3^3

(n_h, n_A, n_π)	(0, 1, 2)	(1, 1, 1)
(r, d, a)	(0, 0, 0)	(1, 0, 0)
Λ_λ^p	Λ_4^4	M_{pl}/m

The most relevant terms are

$$\frac{1}{\Lambda_5^5} (\bar{\nabla}^2 \hat{\pi})^3, \quad \frac{1}{\Lambda_4^4} (\bar{\nabla} \hat{A}) (\bar{\nabla}^2 \hat{\pi})^2,$$

which have cutoff scales lower than Λ_3 . Terms suppressed by Λ_3 are

$$\frac{1}{\Lambda_3^3} \hat{h} (\bar{\nabla}^2 \hat{\pi})^2, \quad \frac{1}{\Lambda_3^3} (\bar{\nabla} \hat{A})^2 \bar{\nabla}^2 \hat{\pi}, \quad \frac{1}{\Lambda_3^3} \frac{\bar{R}}{m^2} \bar{\nabla}^2 \hat{\pi} (\bar{\nabla} \hat{\pi})^2, \quad \frac{1}{\Lambda_3^3} \frac{\bar{\nabla} \bar{R}}{m^2} (\bar{\nabla} \hat{\pi})^3,$$

where the last two types of terms arise due to the presence of curvature of the fiducial metric.

$$\bullet \quad n_h + n_A + n_\pi = 4$$

(n_h, n_A, n_π)	(4, 0, 0)		(0, 4, 0)				(0, 0, 4)				
(r, d, a)	(0, 0, 0)	(0, 0, 0)	(1, 0, 2)	(1, 1, 3)	(1, 2, 4)	(2, 0, 4)	(0, 0, 0)	(1, 0, 0)	(1, 1, 0)	(1, 2, 0)	(2, 0, 0)
Λ_λ^p	M_{pl}^2/m^2	Λ_2^4	M_{pl}^2			M_{pl}^2/m^2	Λ_4^8	Λ_3^6			Λ_2^4
(n_h, n_A, n_π)	(3, 1, 0)	(1, 3, 0)			(3, 0, 1)	(1, 0, 3)					
(r, d, a)	(0, 0, 0)	(0, 0, 0)	(1, 0, 2)	(1, 1, 3)	(0, 0, 0)	(0, 0, 0)	(1, 0, 0)	(1, 1, 0)			
Λ_λ^p	$\Lambda_{1/2}$	$\Lambda_{3/2}^3$	$\Lambda_{1/2}$		M_{pl}^2	Λ_3^6	Λ_2^4				
(n_h, n_A, n_π)	(0, 3, 1)										
(r, d, a)	(0, 0, 0)	(1, 0, 1)	(1, 0, 2)	(1, 1, 2)	(1, 1, 3)	(1, 2, 3)	(2, 0, 3)				
Λ_λ^p	$\Lambda_{5/2}^5$	$\Lambda_{3/2}^3$					$\Lambda_{1/2}$				
(n_h, n_A, n_π)	(0, 1, 3)							(2, 2, 0)		(2, 0, 2)	
(r, d, a)	(0, 0, 0)	(1, 0, 0)	(1, 0, 1)	(1, 1, 0)	(1, 1, 1)	(1, 2, 1)	(2, 0, 1)	(0, 0, 0)	(1, 0, 2)	(0, 0, 0)	(1, 0, 0)
Λ_λ^p	$\Lambda_{7/2}^7$	$\Lambda_{5/2}^5$					$\Lambda_{3/2}^3$	M_{pl}^2	M_{pl}^2/m^2	Λ_2^4	M_{pl}^2
(n_h, n_A, n_π)	(0, 2, 2)								(2, 1, 1)		
(r, d, a)	(0, 0, 0)	(1, 0, 0)	(1, 0, 1)	(1, 0, 2)	(1, 1, 1)	(1, 1, 2)	(1, 2, 2)	(2, 0, 2)	(0, 0, 0)	(1, 0, 1)	
Λ_λ^p	Λ_3^6	Λ_2^4						M_{pl}^2	$\Lambda_{3/2}^3$	$\Lambda_{1/2}$	
(n_h, n_A, n_π)	(1, 2, 1)				(1, 1, 2)						
(r, d, a)	(0, 0, 0)	(1, 0, 1)	(1, 0, 2)	(1, 1, 2)	(0, 0, 0)	(1, 0, 0)	(1, 0, 1)	(1, 1, 1)			
Λ_λ^p	Λ_2^4	M_{pl}^2			$\Lambda_{5/2}^5$	$\Lambda_{3/2}^3$					

In this case, the most relevant terms are

$$\frac{1}{\Lambda_4^8} (\bar{\nabla}^4 \hat{\pi})^4, \quad \frac{1}{\Lambda_{7/2}^7} (\bar{\nabla} \hat{A}) (\bar{\nabla}^4 \hat{\pi})^3,$$

while terms suppressed by Λ_3 are

$$\begin{aligned} & \frac{1}{\Lambda_3^6} \hat{h} (\bar{\nabla}^2 \hat{\pi})^3, \quad \frac{1}{\Lambda_3^6} (\bar{\nabla} \hat{A})^2 (\bar{\nabla}^2 \hat{\pi})^2, \\ & \frac{1}{\Lambda_3^6} \frac{\bar{R}}{m^2} (\bar{\nabla} \hat{\pi})^2 (\bar{\nabla}^2 \hat{\pi})^2, \quad \frac{1}{\Lambda_3^6} \frac{\bar{\nabla} \bar{R}}{m^2} (\bar{\nabla} \hat{\pi})^3 (\bar{\nabla}^2 \hat{\pi}), \quad \frac{1}{\Lambda_3^6} \frac{\bar{\nabla}^2 \bar{R}}{m^2} (\bar{\nabla} \hat{\pi})^4. \end{aligned}$$

Appendix C: EQUATION OF MOTION FOR $\hat{\pi}$ IN THE DECOUPLING LIMIT

First we will show that, as long as the second derivatives of $\hat{\pi}$ enter the Lagrangian linearly, the corresponding equations of motion for $\hat{\pi}$ are up to second order in derivatives. To this end, consider a general Lagrangian

$$\bar{T}_{\lambda_1 \dots \lambda_n \mu \nu} \bar{\nabla}^{\lambda_1} \hat{\pi} \dots \bar{\nabla}^{\lambda_n} \hat{\pi} \bar{\nabla}^\mu \bar{\nabla}^\nu \hat{\pi}, \quad (\text{C1})$$

where $\bar{T}_{\lambda_1 \dots \lambda_n \mu \nu}$ contains no $\hat{\pi}$. Simple manipulation yields the equation of motion, which reads

$$\begin{aligned} 0 = & - \sum_{i=1}^n \bar{\nabla}^{\lambda_i} (T_{\lambda_1 \dots \lambda_i \dots \lambda_n \mu \nu} \bar{\nabla}^{\lambda_1} \hat{\pi} \dots \bar{\nabla}^{\lambda_{i-1}} \hat{\pi} \bar{\nabla}^{\lambda_{i+1}} \hat{\pi} \dots \bar{\nabla}^{\lambda_n} \hat{\pi}) \bar{\nabla}^\mu \bar{\nabla}^\nu \hat{\pi} + \bar{\nabla}^\mu (\bar{\nabla}^\nu T_{\lambda_1 \dots \lambda_n \mu \nu} \bar{\nabla}^{\lambda_1} \hat{\pi} \dots \bar{\nabla}^{\lambda_n} \hat{\pi}) \\ & + \sum_{i=1}^n \bar{\nabla}^\mu (T_{\lambda_1 \dots \lambda_i \dots \lambda_n \mu \nu} \bar{\nabla}^{\lambda_1} \hat{\pi} \dots \bar{\nabla}^{\lambda_{i-1}} \hat{\pi} \bar{\nabla}^{\lambda_{i+1}} \hat{\pi} \dots \bar{\nabla}^{\lambda_n} \hat{\pi}) \bar{\nabla}^\nu \bar{\nabla}^{\lambda_i} \hat{\pi} \\ & + \sum_{i=1}^n T_{\lambda_1 \dots \lambda_i \dots \lambda_n \mu \nu} \bar{\nabla}^{\lambda_1} \hat{\pi} \dots \bar{\nabla}^{\lambda_{i-1}} \hat{\pi} \bar{\nabla}^{\lambda_{i+1}} \hat{\pi} \dots \bar{\nabla}^{\lambda_n} \hat{\pi} \bar{R}^{\mu \lambda_i \nu \rho} \bar{\nabla}_\rho \hat{\pi}, \end{aligned} \quad (\text{C2})$$

and hence contains no higher derivatives of $\hat{\pi}$.

Next, for arbitrary tensors $E_{\mu\nu\rho_1\sigma_1\rho_2\sigma_2}$ and $E_{\mu\nu\rho_1\sigma_1\rho_2\sigma_2\rho_3\sigma_3}$ containing no $\hat{\pi}$, which are antisymmetric under exchange of $\rho_i \leftrightarrow \rho_j$ and $\sigma_i \leftrightarrow \sigma_j$ and symmetric under exchange of pairs of $(\rho_i\sigma_i) \leftrightarrow (\rho_j\sigma_j)$, the corresponding equation of motion for $\hat{\pi}$ of

$$E_{\mu\nu\rho_1\sigma_1\rho_2\sigma_2} \bar{\nabla}^\mu \hat{\pi} \bar{\nabla}^\nu \hat{\pi} \bar{\nabla}^{\rho_1} \bar{\nabla}^{\sigma_1} \hat{\pi} \bar{\nabla}^{\rho_2} \bar{\nabla}^{\sigma_2} \hat{\pi} \quad (C3)$$

and

$$\hat{h}^{\mu\nu} E_{\mu\nu\rho_1\sigma_1\rho_2\sigma_2\rho_3\sigma_3} \bar{\nabla}^{\rho_1} \bar{\nabla}^{\sigma_1} \hat{\pi} \bar{\nabla}^{\rho_2} \bar{\nabla}^{\sigma_2} \hat{\pi} \bar{\nabla}^{\rho_3} \bar{\nabla}^{\sigma_3} \hat{\pi}, \quad (C4)$$

are

$$\begin{aligned} 0 = & -2\bar{\nabla}^\mu (E_{\mu\nu\rho_1\sigma_1\rho_2\sigma_2} \bar{\nabla}^\nu \hat{\pi}) \bar{\nabla}^{\rho_1} \bar{\nabla}^{\sigma_1} \hat{\pi} \bar{\nabla}^{\rho_2} \bar{\nabla}^{\sigma_2} \hat{\pi} + 4E_{\mu\nu\rho_1\sigma_1\rho_2\sigma_2} \bar{R}^{\rho_1\mu\sigma_1\lambda} \bar{\nabla}_\lambda \hat{\pi} \bar{\nabla}^\nu \hat{\pi} \bar{\nabla}^{\rho_2} \bar{\nabla}^{\sigma_2} \hat{\pi} \\ & + (\bar{\nabla}^{\sigma_1} E_{\mu\nu\rho_1\sigma_1\rho_2\sigma_2} \bar{\nabla}^\mu \hat{\pi} + 2E_{\mu\nu\rho_1\sigma_1\rho_2\sigma_2} \bar{\nabla}^{\sigma_1} \bar{\nabla}^\mu \hat{\pi}) \bar{\nabla}^\nu \hat{\pi} \bar{R}^{\rho_1\rho_2\sigma_2\lambda} \bar{\nabla}_\lambda \hat{\pi} \\ & + 2\bar{\nabla}^{\rho_1} (\bar{\nabla}^{\sigma_1} E_{\mu\nu\rho_1\sigma_1\rho_2\sigma_2} \bar{\nabla}^\mu \hat{\pi} \bar{\nabla}^\nu \hat{\pi}) \bar{\nabla}^{\rho_2} \bar{\nabla}^{\sigma_2} \hat{\pi} + 4\bar{\nabla}^{\rho_1} (E_{\mu\nu\rho_1\sigma_1\rho_2\sigma_2} \bar{\nabla}^\nu \hat{\pi}) \bar{\nabla}^{\sigma_1} \bar{\nabla}^\mu \hat{\pi} \bar{\nabla}^{\rho_2} \bar{\nabla}^{\sigma_2} \hat{\pi} \\ & + \bar{\nabla}^{\rho_1} (E_{\mu\nu\rho_1\sigma_1\rho_2\sigma_2} \bar{\nabla}^\mu \hat{\pi} \bar{\nabla}^\nu \hat{\pi} \bar{R}^{\sigma_1\sigma_2\rho_2\lambda} \bar{\nabla}_\lambda \hat{\pi}), \end{aligned} \quad (C5)$$

and

$$0 = \hat{h}^{\mu\nu} E_{\mu\nu\rho_1\sigma_1\rho_2\sigma_2\rho_3\sigma_3} [\bar{\nabla}^{\rho_1} (\bar{R}^{\sigma_1\sigma_2\rho_2\lambda} \bar{\nabla}_\lambda \hat{\pi}) \bar{\nabla}^{\rho_2} \bar{\nabla}^{\sigma_3} \hat{\pi} + \bar{R}^{\sigma_1\sigma_2\rho_2\lambda} \bar{R}^{\rho_1\rho_2\sigma_3\tau} \bar{\nabla}_\lambda \hat{\pi} \bar{\nabla}_\tau \hat{\pi}], \quad (C6)$$

respectively, which also contain no higher derivatives of $\hat{\pi}$. Terms in \mathcal{L}_4 are just special cases of the above.

Appendix D: USEFUL INTEGRATION BY PARTS

In the case of a flat fiducial metric, the self-interactions of the helicity-0 mode π exactly vanish because they become total derivatives. In the case of a curved fiducial metric, instead we have, at the quadratic order

$$(\Box\pi)^2 - \bar{\nabla}_\mu \bar{\nabla}_\nu \pi \bar{\nabla}^\mu \bar{\nabla}^\nu \pi \simeq \bar{R}_{\mu\nu} \bar{\nabla}^\mu \pi \bar{\nabla}^\nu \pi, \quad (D1)$$

at the cubic order

$$\begin{aligned} & (\Box\pi)^3 - 3\Box\pi \bar{\nabla}_\mu \bar{\nabla}_\nu \pi \bar{\nabla}^\mu \bar{\nabla}^\nu \pi + 2\bar{\nabla}_\mu \bar{\nabla}^\nu \pi \bar{\nabla}_\nu \bar{\nabla}^\rho \pi \bar{\nabla}_\rho \bar{\nabla}^\mu \pi \\ & \simeq 2\bar{R}_{\mu\nu} \bar{\nabla}^\nu \pi (\bar{\nabla}^\mu \pi \Box\pi - \bar{\nabla}^\mu \bar{\nabla}_\rho \pi \bar{\nabla}^\rho \pi) - 2\bar{R}_{\mu\rho\nu\sigma} \bar{\nabla}^\rho \pi \bar{\nabla}^\sigma \pi \bar{\nabla}^\mu \bar{\nabla}^\nu \pi, \end{aligned} \quad (D2)$$

and at the quartic order

$$\begin{aligned} & (\Box\pi)^4 - 6(\Box\pi)^2 \bar{\nabla}_\mu \bar{\nabla}_\nu \pi \bar{\nabla}^\mu \bar{\nabla}^\nu \pi + 8\Box\pi \bar{\nabla}^\mu \bar{\nabla}_\nu \pi \bar{\nabla}^\nu \bar{\nabla}_\rho \pi \bar{\nabla}^\rho \bar{\nabla}_\mu \pi \\ & + 3(\bar{\nabla}_\mu \bar{\nabla}_\nu \pi \bar{\nabla}^\mu \bar{\nabla}^\nu \pi)^2 - 6\bar{\nabla}^\mu \bar{\nabla}_\nu \pi \bar{\nabla}^\nu \bar{\nabla}_\rho \pi \bar{\nabla}^\rho \bar{\nabla}_\sigma \pi \bar{\nabla}^\sigma \bar{\nabla}_\mu \pi \\ & \simeq 3\bar{R}_{\mu\nu} \bar{\nabla}^\mu \pi \bar{\nabla}^\nu \pi [(\Box\pi)^2 - \bar{\nabla}_\rho \bar{\nabla}_\sigma \pi \bar{\nabla}^\rho \bar{\nabla}^\sigma \pi] - 6\bar{R}_{\mu\nu} \bar{\nabla}^\alpha \pi \bar{\nabla}^\mu \pi (\bar{\nabla}^\nu \bar{\nabla}_\alpha \pi \Box\pi - \bar{\nabla}^\rho \bar{\nabla}_\alpha \pi \bar{\nabla}^\nu \bar{\nabla}_\rho \pi) \\ & - 6\bar{R}_{\mu\rho\nu\sigma} [\bar{\nabla}^\mu \pi \bar{\nabla}^\nu \pi (\Box\pi \bar{\nabla}^\rho \bar{\nabla}^\sigma \pi - \bar{\nabla}^\lambda \bar{\nabla}^\rho \pi \bar{\nabla}^\sigma \bar{\nabla}_\lambda \pi) - \bar{\nabla}^\alpha \pi \bar{\nabla}^\mu \pi \bar{\nabla}^\nu \bar{\nabla}_\alpha \pi \bar{\nabla}^\rho \bar{\nabla}^\sigma \pi]. \end{aligned} \quad (D3)$$

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